Chapter 10 Topics in Analytic Geometry

Course/Section
Lesson Number
Date

Section 10.5 Rotation of Conics

Section Objectives: Students will know how to eliminate the *xy*-term in the equation of a conic and use the discriminant to identify a conic.

I. Rotation (pp. 763−766) Pace: 20 minutes

- Remind students that we now know how to graph conics whose equations are of the form $Ax^2 + Cy^2 + Dx + Ey + F = 0$.

- State that in this section, we look at graphing conics whose equations are of the form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$. Note that because of the *xy*-term in the equation, we cannot complete the square. Furthermore, conics with equations of this type have axes that are rotated and not parallel to either the *x*-axis or the *y*-axis.

- State that to eliminate the *xy*-term in the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, we write it in the form $A'(x')^2 + C'(y')^2 + D'x + E'y + F' = 0$ by rotating the coordinate axes through an angle θ, where $\cot 2\theta = \frac{A-C}{B}$. The coefficients of the new equation are obtained by making the substitutions $x = x'\cos\theta - y'\sin\theta$ and $y = x'\sin\theta + y'\cos\theta$.

Example 1. Write the following equation in standard form.

$$xy = \frac{1}{2}$$

$$\cot 2\theta = 0 \Rightarrow \theta = \frac{\pi}{4}$$

$$x = x'\cos\frac{\pi}{4} - y'\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}x' - \frac{\sqrt{2}}{2}y' = \frac{x'-y'}{\sqrt{2}}$$

$$y = x'\sin\frac{\pi}{4} + y'\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y' = \frac{x'+y'}{\sqrt{2}}$$

Make these substitutions into the equation.

$$\frac{x'-y'}{\sqrt{2}} \cdot \frac{x'+y'}{\sqrt{2}} = \frac{1}{2}$$

$$(x')^2 - (y')^2 = 1$$

Example 2. Sketch the graph of $x^2 + \sqrt{3}xy + 2y^2 - 2 = 0$.

$$\cot 2\theta = \frac{1-2}{\sqrt{3}} = -\frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{\pi}{3}$$

$$x = x'\cos\frac{\pi}{3} - y'\sin\frac{\pi}{3} = \frac{1}{2}x' - \frac{\sqrt{3}}{2}y'$$

$$y = x'\sin\frac{\pi}{3} + y'\cos\frac{\pi}{3} = \frac{\sqrt{3}}{2}x' + \frac{1}{2}y'$$

$$\left(\frac{1}{2}x' - \frac{\sqrt{3}}{2}y'\right)^2 + \sqrt{3}\left(\frac{1}{2}x' - \frac{\sqrt{3}}{2}y'\right)\left(\frac{\sqrt{3}}{2}x' + \frac{1}{2}y'\right)$$

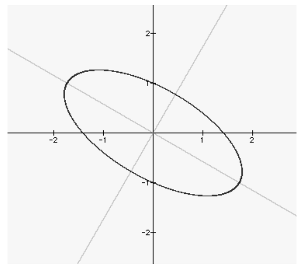
$$+ 2\left(\frac{\sqrt{3}}{2}x' + \frac{1}{2}y'\right)^2 - 2 = 0$$

which simplifies to

$$\frac{5}{2}(x')^2 + \frac{(y')^2}{2} = 2$$

$$\frac{(x')^2}{4/5} + \frac{(y')^2}{4} = 1$$

So, the vertices are at (0, ±2) and the covertices are at approximately (±0.89, 0) on the rotated axes.



Example 3. Sketch the graph of $3x^2 + 2\sqrt{3}xy + y^2 + 2x - 2\sqrt{3}y = 0$.

$$\cot 2\theta = \frac{3-1}{2\sqrt{3}} = \frac{\sqrt{3}}{3} \Rightarrow 2\theta = \frac{\pi}{3} \Rightarrow \theta = \frac{\pi}{6}$$

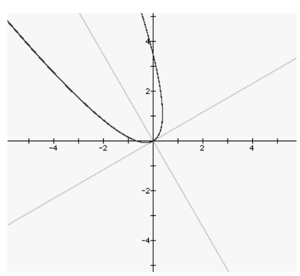
$$x = x'\cos\frac{\pi}{6} - y'\sin\frac{\pi}{6} = \frac{\sqrt{3}x' - y'}{2}$$

$$y = x'\sin\frac{\pi}{6} + y'\cos\frac{\pi}{6} = \frac{x' + \sqrt{3}y'}{2}$$

$$3\left(\frac{\sqrt{3}x' - y'}{2}\right)^2 + 2\sqrt{3}\left(\frac{\sqrt{3}x' - y'}{2}\right)\left(\frac{x' + \sqrt{3}y'}{2}\right) + \left(\frac{x' + \sqrt{3}y'}{2}\right)^2$$

$$+2\left(\frac{\sqrt{3}x' - y'}{2}\right) - 2\sqrt{3}\left(\frac{x' + \sqrt{3}y'}{2}\right) = 0$$

$$4(x')^2 - 4y' = 0 \Rightarrow y' = (x')^2$$



II. Invariants Under Rotation (pp. 767–768) Pace: 10 minutes

- State that the quantities in the equation that do not change when rotating the axes are called the **invariants under rotation.** Using the methods of this section, the invariants under rotation are:
  1. $F = F'$
  2. $A + C = A' + C'$
  3. $B^2 - 4AC = (B')^2 - 4A'C'$

- State that since we rotate the axes to get B' to be 0, we have $B^2 - 4AC = -4A'C'$, called the **discriminant.**

- State that the discriminant can be used to classify the conic before rotation as follows: The graph of $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ is
  1. an ellipse or a circle if $B^2 - 4AC < 0$,
  2. a parabola if $B^2 - 4AC = 0$, or
  3. a hyperbola if $B^2 - 4AC > 0$.

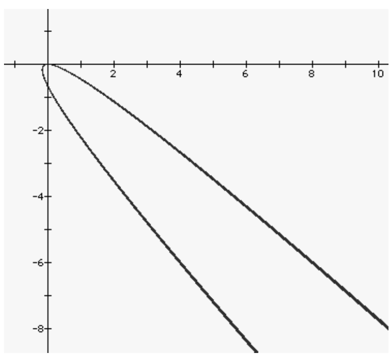
Example 4. For each of the following, classify the graph, use the quadratic formula to solve for y, and use a graphing utility to graph the equation.

a) $3x^2 - 6xy + 3y^2 + 2y = 0$

$(-6)^2 - 4(3)(3) = 0$. Thus the conic is a parabola.

$3x^2 - 6xy + 3y^2 + 2y = 3y^2 + (2 - 6x)y + 3x^2 = 0$

$$y = \frac{-(2-6x) \pm \sqrt{(2-6x)^2 - 4(3)(3x^2)}}{2(3)}$$

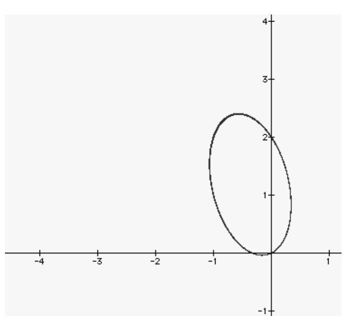


b) $3x^2 - xy + y^2 + x - 2y = 0$

$1^2 - 4(3)(1) = -11 < 0$. Thus the conic is an ellipse.

$3x^2 - xy + y^2 + x - 2y = y^2 + (-x - 2) + (3x^2 + x) = 0$

$$y = \frac{-(-x-2) \pm \sqrt{(-x-2)^2 - 4(1)(3x^2 + x)}}{2(1)}$$



c) $x^2 + 5xy + 2y^2 + 4x = 0$

$5^2 - 4(1)(2) = 17 > 0$. Thus the conic is a hyperbola.

$$x^2 + 5xy + 2y^2 + 4x = 2y^2 + (5x)y + (x^2 + 4x) = 0$$

$$y = \frac{-5x \pm \sqrt{(5x)^2 - 4(2)(x^2 + 4x)}}{2(2)}$$