Chapter 11 Analytic Geometry in Three Dimensions

Course/Section Lesson Number Date

Section 11.2 Vectors in Space

Section Objectives: Students will know how to represent vectors in space, how to find dot products of and angles between vectors in space, and how to determine if vectors in space are parallel or orthogonal.

I. Vectors in Space (pp. 820−822) Pace: 15 minutes

- State that in space vectors are represented by ordered triples $\mathbf{v} = \langle v_1, v_2, v_3\rangle$. They can also be written in **standard unit vector notation** $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, where $\mathbf{i} = \langle 1, 0, 0\rangle$, $\mathbf{j} = \langle 0, 1, 0\rangle$, and $\mathbf{k} = \langle 0, 0, 1\rangle$.

- State that the **zero vector** is denoted by $\mathbf{0} = \langle 0, 0, 0\rangle$.

- State that the **component form** of the vector from the point $P(p_1, p_2, p_3)$ to the point $Q(q_1, q_2, q_3)$ is $\mathbf{v} = \langle v_1, v_2, v_3\rangle = \langle q_1 - p_1, q_2 - p_2, q_3 - p_3\rangle$.

- State the following definitions.
  1. Two vectors are equal if and only if their corresponding components are equal.
  2. The **magnitude** of $\mathbf{v} = \langle v_1, v_2, v_3\rangle$ is $\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$.
  3. If $\mathbf{v} \neq \mathbf{0}$, then the unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v}$.
  4. Let $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3\rangle$. Then $\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3\rangle$.
  5. $k\mathbf{u} = k\langle u_1, u_2, u_3\rangle = \langle ku_1, ku_2, ku_3\rangle$.
  6. The **dot product** of vectors $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3\rangle$ is $\mathbf{u} \bullet \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$.

  Example 1. Find the component form of the vector from (2, –4, 1) to (5, 7, 0). Also find the unit vector in the direction of **v**.
  $\mathbf{v} = \langle 5 - 2, 7 - (-4), 0 - 1\rangle = \langle 3, 11, -1\rangle$.
  $\|\mathbf{v}\| = \sqrt{3^2 + 11^2 + (-1)^2} = \sqrt{131}$
  $\mathbf{u} = \left\langle \frac{3}{\sqrt{131}}, \frac{11}{\sqrt{131}}, \frac{-1}{\sqrt{131}} \right\rangle$

  Example 2. Let $\mathbf{u} = \langle 2, 6, 2\rangle$ and $\mathbf{v} = \langle -1, 5, 0\rangle$. Find the dot product.
  $\mathbf{u} \bullet \mathbf{v} = \langle 2, 6, 2\rangle \bullet \langle -1, 5, 0\rangle = 2(-1) + 6(5) + 2(0) = 28$

- State that the **angle between two nonzero vectors** is the angle θ, $0 \leq \theta \leq \pi$, between their respective standard position vectors.

- State that the angle θ between two nonzero vectors **u** and **v** can be found from $\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$.

Example 3. Find the angle between **u** = ⟨–2, 3, 0⟩ and **v** = ⟨1, –5, 1⟩.

$$\theta = \cos^{-1}\frac{\langle -2,3,0\rangle\cdot\langle 1,-5,1\rangle}{\|\langle -2,3,0\rangle\|\|\langle 1,-5,1\rangle\|} = \cos^{-1}\left(-\frac{17}{3\sqrt{39}}\right) = 155.1^\circ$$

II. Parallel Vectors (pp. 822–823) Pace: 10 minutes

- State that two nonzero vectors **u** and **v** are parallel if and only if there is some scalar c such that **u** = c**v**.

Example 4. Are the vectors **u** = ⟨2, 6, 1⟩ and **v** = ⟨10, 30, 5⟩ parallel?
Yes, since **v** = 5**u.**

Example 5. Are the points $P(1, 3, 4)$, $Q(3, 2, 7)$, and $R(5, 1, 10)$ collinear?
We want to see if vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are parallel.
$\overrightarrow{PQ} = \langle 3-1, 2-3, 7-4\rangle = \langle 2, -1, 3\rangle$
$\overrightarrow{PR} = \langle 5-1, 1-3, 10-4\rangle = \langle 4, -2, 6\rangle$
$\overrightarrow{PR} = 2\overrightarrow{PQ}$
Yes the three points are collinear.

Example 6. The terminal point of the vector **v** = ⟨1, 3, 5⟩ is $Q(2, 5, 8)$. What is the initial point?
$\overrightarrow{PQ} = \langle 2-p_1, 5-p_2, 8-p_3\rangle = \langle 1, 3, 5\rangle$. So, $2-p_1 = 1$, $5-p_2 = 3$, $8-p_3 = 5$, and $p_1 = 1$, $p_2 = 2$, $p_3 = 3$. Hence the initial point is $P(1, 2, 3)$.

III. Application (p. 824) Pace: 5 minutes

Example 7. A 500-pound weight is supported by three ropes. The weight is located at the origin and the ropes are tied to the points $P(2, 0, 2)$, $Q(2, 2, 2)$, and $R(-2, -1, 1)$. Find the tension in each rope.

w = ⟨0, 0, –500⟩ represents the weight. The ropes are represented as follows.

$$\mathbf{u} = \|\mathbf{u}\|\frac{\overrightarrow{OP}}{\|\overrightarrow{OP}\|} = \|\mathbf{u}\|\frac{\langle 2,0,2\rangle}{2\sqrt{2}} = \|\mathbf{u}\|\left\langle \frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right\rangle$$

$$\mathbf{v} = \|\mathbf{v}\|\frac{\overrightarrow{OQ}}{\|\overrightarrow{OQ}\|} = \|\mathbf{v}\|\frac{\langle 2,2,2\rangle}{2\sqrt{3}} = \|\mathbf{v}\|\left\langle \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right\rangle$$

$$\mathbf{z} = \|\mathbf{z}\|\frac{\overrightarrow{OR}}{\|\overrightarrow{OR}\|} = \|\mathbf{z}\|\frac{\langle -2,-1,1\rangle}{\sqrt{6}} = \|\mathbf{z}\|\left\langle -\frac{\sqrt{6}}{3}, -\frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{6}\right\rangle$$

u + **v** + **z** = **–w.** This yields

$$\begin{cases} \frac{\sqrt{2}}{2}\|\mathbf{u}\| + \frac{\sqrt{3}}{3}\|\mathbf{v}\| - \frac{\sqrt{6}}{3}\|\mathbf{z}\| = 0 \\ \frac{\sqrt{3}}{3}\|\mathbf{v}\| - \frac{\sqrt{6}}{6}\|\mathbf{z}\| = 0 \\ \frac{\sqrt{2}}{2}\|\mathbf{u}\| + \frac{\sqrt{3}}{3}\|\mathbf{v}\| + \frac{\sqrt{6}}{6}\|\mathbf{z}\| = 500 \end{cases}$$

Solving the system of equations yields
$\|\mathbf{u}\| \approx 235.7$
$\|\mathbf{v}\| \approx 288.7$
$\|\mathbf{z}\| \approx 408.2$