Chapter 11 Analytic Geometry in Three Dimensions

Course/Section
Lesson Number
Date

Section 11.3 The Cross Product of Two Vectors

Section Objectives: Students will know how to find the cross products of vectors in space, use geometric properties of the cross product, and use triple scalar products to find volumes of parallelepipeds.

I. The Cross Product (pp. 827−828) Pace: 10 minutes

- State that we now look at a product of two vectors that is a vector, unlike the dot product.

- State that the **cross product** of $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ is the vector $\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$. As a memory aid we use the *determinant form* $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$.

- Discuss the *Exploration* on page 827 of the text.

Example 1. Given $\mathbf{u} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\mathbf{v} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$, find the following.

a)

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ 3 & -1 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix}\mathbf{k}$$

$$= 3\mathbf{i} - 5\mathbf{j} - 7\mathbf{k}$$

b)

$$\mathbf{v} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 2 \\ 1 & 2 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 2 \\ 2 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & 2 \\ 1 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & -1 \\ 1 & 2 \end{vmatrix}\mathbf{k}$$

$$= -3\mathbf{i} + 5\mathbf{j} + 7\mathbf{k}$$

c)

$$\mathbf{u} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ 1 & 2 & -1 \end{vmatrix} = 0$$

- State the following **properties of the cross product.**
  1. $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$
  2. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$
  3. $c(\mathbf{u} \times \mathbf{v}) = (c\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (c\mathbf{v})$
  4. $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$
  5. $\mathbf{u} \times \mathbf{u} = \mathbf{0}$
  6. $\mathbf{u} \bullet (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \bullet \mathbf{w}$

II. Geometric Properties of the Cross Product (pp. 829–830)
Pace: 15 minutes

- State the following **geometric properties of the cross product**. Let **u** and **v** be nonzero vectors in space, and let θ be the angle between **u** and **v**.
  1. $\mathbf{u} \times \mathbf{v}$ is orthogonal to both **u** and **v**.
  2. $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\,\|\mathbf{v}\| \sin \theta$
  3. $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ if and only if **u** and **v** are scalar multiples of each other.
  4. $\|\mathbf{u} \times \mathbf{v}\|$ is the area of the parallelogram with **u** and **v** as adjacent sides.

Example 2. Find a unit vector that is orthogonal to both $\mathbf{u} = \mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$ and $\mathbf{v} = 4\mathbf{j} - 5\mathbf{k}$.

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & -3 \\ 0 & 4 & -5 \end{vmatrix} = 22\mathbf{i} + 5\mathbf{j} + 4\mathbf{k}$$

$$\|\mathbf{u} \times \mathbf{v}\| = \sqrt{22^2 + 5^2 + 4^2} = 5\sqrt{21}$$

So the unit vector is

$$\frac{\mathbf{u} \times \mathbf{v}}{\|\mathbf{u} \times \mathbf{v}\|} = \frac{22}{5\sqrt{21}}\mathbf{i} + \frac{5}{5\sqrt{21}}\mathbf{j} + \frac{4}{5\sqrt{21}}\mathbf{k}$$

$$= \frac{22\sqrt{21}}{105}\mathbf{i} + \frac{\sqrt{21}}{21}\mathbf{j} + \frac{4\sqrt{21}}{105}\mathbf{k}$$

Example 3. Find the area of the parallelogram with vertices $P(1, 1, 1)$, $Q(2, 3, -1)$, $R(-1, -2, 2)$, and $S(0, 0, 0)$.
Adjacent sides would be $\overrightarrow{PQ} = \langle 1, 2, -2 \rangle$ and $\overrightarrow{PR} = \langle -2, -3, 1 \rangle$. So, the area is $\|\overrightarrow{PQ} \times \overrightarrow{PR}\| = \|\langle -4, -5, 1 \rangle\| = \sqrt{(-4)^2 + (-5)^2 + 1^2} = \sqrt{42}$.

III. The Triple Scalar Product (p. 831) Pace: 5 minutes

- State that the **triple scalar product** of **u**, **v**, and **w** is
$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

- State that the volume V of a parallelepiped with vectors **u**, **v**, and **w** as adjacent edges is $V = |\mathbf{u} \bullet (\mathbf{v} \times \mathbf{w})|$.

Example 4. Find the volume of the parallelepiped with adjacent edges $\mathbf{u} = 3\mathbf{i} + \mathbf{j} - \mathbf{k}$, $\mathbf{v} = \mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$, and $\mathbf{w} = 2\mathbf{i} - \mathbf{j}$.

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 3 & 1 & -1 \\ 1 & 3 & -5 \\ 2 & -1 & 0 \end{vmatrix} = -18$$

$$V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = |-18| = 18$$