Chapter 11 Analytic Geometry in Three Dimensions

Course/Section
Lesson Number
Date

Section 11.4 Lines and Planes in Space

Section Objectives: Students will know how to find parametric and symmetric equations of lines in space, how to sketch and find equations of planes in space, and how to find the distance between points and planes in space.

I. Lines in Space (pp. 834−835) Pace: 10 minutes

- State that the line L that passes through the point $P(x_1, x_2, x_3)$ and is parallel to the vector $\mathbf{v} = \langle a, b, c\rangle$ is the set of all points $Q(x, y, z)$ for which the vector from P to Q is parallel to $\mathbf{v}$. This implies

$$\overrightarrow{PQ} = t\mathbf{v}$$

$$\langle x - x_1, y - y_1, z - z_1\rangle = \langle ta, tb, tc\rangle$$

Hence the parametric equations for this line are

$$x = x_1 + at \qquad y = y_1 + bt \qquad z = z_1 + ct,$$

and the symmetric equations are $\dfrac{x - x_1}{a} = \dfrac{y - y_1}{b} = \dfrac{z - z_1}{c}$.

- State that in the above equations, the vector $\mathbf{v}$ is the **direction vector** for the line L, and a, b, and c are the **direction numbers.**

Example 1. Find the parametric and symmetric equations of the line passing through the point (1, 0, 5) and parallel to the vector ⟨2, 8, 1⟩.
$x_1 = 1$, $y_1 = 0$, $z_1 = 5$, $a = 2$, $b = 8$, and $c = 1$. Hence,

$$x = 1 + 2t \qquad y = 8t \qquad z = 5 + t$$

and $\dfrac{x-1}{2} = \dfrac{y}{8} = z - 5$.

Example 2. Find the parametric equations of the line passing through the points (–1, 2, 4) and (2, 6, 1).
$\mathbf{v} = \langle 2 - (-1), 6 - 2, 1 - 4\rangle = \langle 3, 4, -3\rangle$
$x = 2 + 3t, \qquad y = 6 + 4t, \qquad z = 1 - 3t.$

II. Planes in Space (pp. 836−838) Pace: 15 minutes

- State that the plane containing the point $P(x_1, x_2, x_3)$ and having nonzero normal vector $\mathbf{n} = \langle a, b, c\rangle$ is the set of all points $Q(x, y, z)$ for which the vector from P to Q is orthogonal to $\mathbf{n}$. This yields

$$\mathbf{n} \cdot \overrightarrow{PQ} = 0$$

$$\langle a, b, c\rangle \cdot \langle x - x_1, y - y_1, z - z_1\rangle = 0$$

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

- State that this is the **standard form of the equation of a plane** and that the **general form** is $ax + by + cz + d = 0$.

Example 3. Find the general form of the equation of the plane that contains the points (–1, 2, 5), (2, 0, 1), and (3, 3, 4).
n will be the cross product of the vectors
$\mathbf{u} = \langle 2-(-1), 0-2, 1-5\rangle = \langle 3, -2, -4\rangle$ and $\mathbf{v} = \langle 3-(-1), 3-2, 4-5\rangle$

$$= \langle 4, 1, -1\rangle. \text{ So, } \mathbf{n} = \mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & -4 \\ 4 & 1 & -1 \end{vmatrix} = \langle 6, -13, 11\rangle.$$ Using this vector and the point (2, 0, 1), the equation of the plane is

$$6(x-2)-13(y-0)+11(z-1)=0$$
$$6x-13y-11z-23=0.$$

- State that to find the angle between two planes, we find the angle between their corresponding normal vectors.

Example 4. Find the angle between the two planes given by the following equations. Also find the parametric equations of their line of intersection.

$$2x + 3y - 2z = 0$$
$$x - y + 3z = 0$$

The normal vectors are $\mathbf{n}_1 = \langle 2, 3, -2\rangle$ and $\mathbf{n}_2 = \langle 1, -1, 3\rangle$. The angle between these two vectors is found from

$$\cos\theta = \frac{|\langle 2,3,-2\rangle\cdot\langle 1,-1,3\rangle|}{\|\langle 2,3,-2\rangle\|\|\langle 1,-1,3\rangle\|} = \frac{7}{\sqrt{17}\sqrt{11}} = \frac{7}{\sqrt{187}}.$$

Hence $\theta \approx 59.21°$. To find the parametric equations of the line of intersection, find the cross product of the two normal vectors to find the direction vector, and use the point (0, 0, 0).
$\mathbf{n}_1 \times \mathbf{n}_2 = \langle 7, -8, -5\rangle$. So, the equations are
$x = 7t$, $y = -8t$, $z = -5t$.

III. Sketching Planes in Space (p. 839) Pace: 5 minutes

- State that to sketch a plane in space, it is helpful to sketch the traces in all coordinate planes through which it passes.
- State that if a variable is missing from the equation of a plane, then the plane is parallel to *that* axis.

IV. Distance Between a Point and a Plane (p. 840) Pace: 5 minutes

- State that the distance D between a point Q and a plane is found by taking any point P on the plane, projecting the vector from P to Q onto the normal vector **n**, and finding the magnitude of this projection. Hence,

$$D = \|\mathbf{proj}_{\mathbf{n}}\overrightarrow{PQ}\| = \frac{|\overrightarrow{PQ}\cdot\mathbf{n}|}{\|\mathbf{n}\|}.$$

Example 5. Find the distance between the point $Q(1, -1\ 2)$ and the plane $x - y + 2z = 5$.
We will use the point $P(5, 0, 0)$ as the point on the plane.
$\overrightarrow{PQ} = \langle 1-5, -1-0, 2-0\rangle = \langle -4, -1, 2\rangle$. Since $\mathbf{n} = \langle 1, -1, -2\rangle$,

$$D = \frac{|\langle -4,-1,2\rangle\cdot\langle 1,-1,-2\rangle|}{\sqrt{1+1+4}} = \frac{|-7|}{\sqrt{6}} = \frac{7\sqrt{6}}{6}.$$