Chapter 12 Limits and an Introduction to Calculus

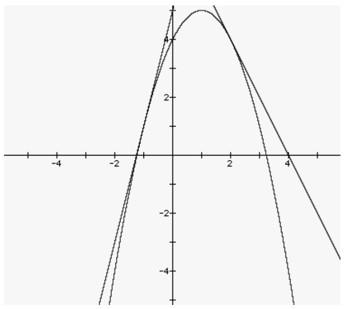
Section 12.3 The Tangent Line Problem

Course/Section Lesson Number Date

Section Objectives: Students will know how to approximate slopes of tangent lines, use the limit definition of slope, and use derivatives to find slope.

I. Tangent Line to a Graph (p. 873) Pace: 5 minutes

- Draw a picture similar to the one below. Discuss how tangent lines can be used to describe the rate at which a curve rises or falls.

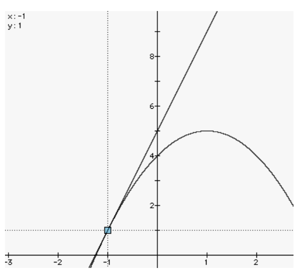


- State that, unlike with circles, tangent lines can meet curves in more than one point. In addition, a tangent line can be thought of as the line that best approximates the curve at the point of tangency.

II. Slope of a Graph (p. 874) Pace: 5 minutes

- State that we call the tangent line to a curve at a particular point the **slope** of the graph at that point.

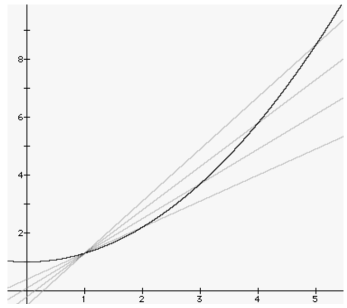
Example 1. Approximate the slope of the graph at (−1, 1).



It appears that if, from (−1, 1), we go up 4 and to the right 1, we will land on another point on the line. Hence the slope is 4/1 = 4.

III. Slope and the Limit Process (pp. 875–877) Pace: 15 minutes

- Draw a graph similar to the one below. Point out how the secant lines approach the tangent line. Hence the slope of the secant lines must approach the slope of the tangent line.



- State that since the slope of the secant line is $\frac{f(x+h)-f(x)}{h}$, the slope of the tangent line to the graph of $y = f(x)$ at $(x, f(x))$ is
$$m = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$
provided the limit exists.

Example 2. Find the slope of the graph of $y = x^2 + 1$ at (1, 2).

$$m = \lim_{h \to 0} \frac{\left[(1+h)^2 + 1\right] - \left[1^2 + 1\right]}{h}$$
$$= \lim_{h \to 0} \frac{1 + 2h + h^2 + 1 - 2}{h}$$
$$= \lim_{h \to 0} \frac{2h + h^2}{h}$$
$$= \lim_{h \to 0} (2 + h)$$
$$= 2$$

Example 3. Find the slope of the graph of $f(x) = 2x^2 - 3$ at $(x, f(x))$.

$$m = \lim_{h \to 0} \frac{\left[2(x+h)^2 - 3\right] - \left[2x^2 - 3\right]}{h}$$
$$= \lim_{h \to 0} \frac{\left[2(x^2 + 2xh + h^2) - 3\right] - \left[2x^2 - 3\right]}{h}$$
$$= \lim_{h \to 0} \frac{4xh + 2h^2}{h}$$
$$= \lim_{h \to 0} (4x + 2h)$$
$$= 4x$$

IV. The Derivative of a Function (pp. 878–879) Pace: 10 minutes

- State that our result in the above example is itself a function, called the **derivative** of $f(x) = 2x^2 - 3$, denoted by $f'(x) = 4x$.
- State that the derivative of a function $f(x)$ is $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$, provided the limit exists.
- Discuss the various notations for the derivative:
  $\frac{dy}{dx}$, y', $\frac{d}{dx}[f(x)]$, and $D_x[y]$.

Example 4. Find the derivative of the following functions.

a)

$$f(x) = x^2 + x - 2$$

$$f'(x) = \lim_{h\to 0} \frac{\left[(x+h)^2 + (x+h) - 2\right] - \left[x^2 + x - 2\right]}{h}$$

$$= \lim_{h\to 0} \frac{\left[\left(x^2 + 2xh + h^2\right) + (x+h) - 2\right] - \left[x^2 + x - 2\right]}{h}$$

$$= \lim_{h\to 0} \frac{2xh + h^2 + h}{h}$$

$$= \lim_{h\to 0} (2x + h + 1)$$

$$= 2x + 1$$

b)

$$f(x) = \frac{1}{x}$$

$$f'(x) = \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \cdot \frac{x(x+h)}{x(x+h)}$$

$$= \lim_{h\to 0} \frac{x - (x+h)}{hx(x+h)}$$

$$= \lim_{h\to 0} \frac{-h}{hx(x+h)}$$

$$= \lim_{h\to 0} \frac{-1}{x(x+h)}$$

$$= -\frac{1}{x^2}$$