Chapter 12 Limits and an Introduction to Calculus

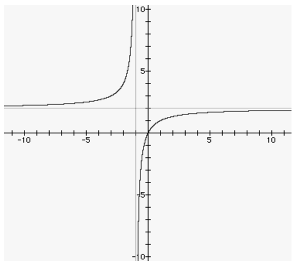
Course/Section
Lesson Number
Date

Section 12.4 Limits at Infinity and Limits of Sequences

Section Objectives: Students will know how to evaluate limits at infinity and find limits of sequences.

I. Limits at Infinity and Horizontal Asymptotes (pp. 883−886)
Pace: 15 minutes

- Draw the graph of $f(x) = 2x/(x + 1)$. State that we know that the horizontal asymptote is the line $y = 2$. Now we say $\lim_{x\to\infty} f(x) = 2$, and $\lim_{x\to-\infty} f(x) = 2$.



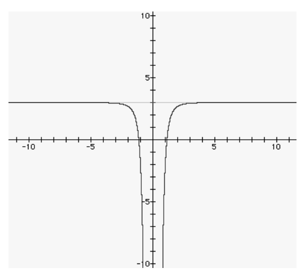
- State the following facts about **limits at infinity.** If r is a positive real number, then $\lim_{x\to\infty} \frac{1}{x^r} = 0$. Furthermore, if x^r is defined for $x < 0$, then $\lim_{x\to-\infty} \frac{1}{x^r} = 0$.

Example 1. Evaluate $\lim_{x\to\infty} \frac{3x^4 - 5}{x^4}$.

Algebraic Solution

$$\lim_{x\to\infty} \frac{3x^4 - 5}{x^4} = \lim_{x\to\infty}\left(3 - \frac{5}{x^4}\right)$$
$$= \lim_{x\to\infty} 3 - \lim_{x\to\infty} \frac{5}{x^4}$$
$$= 3 - 0$$
$$= 3$$

Graphical Solution



Note that the horizontal asymptote is $y = 3$.

Example 2. Find the limits of the following functions as x approaches infinity.

a)

$$\lim_{x\to\infty} \frac{x-1}{x^2} = \lim_{x\to\infty} \frac{\frac{x}{x^2} - \frac{1}{x^2}}{\frac{x^2}{x^2}}$$
$$= \lim_{x\to\infty} \frac{\frac{1}{x} - \frac{1}{x^2}}{1}$$
$$= \frac{0 - 0}{1}$$
$$= 0$$

b)

$$\lim_{x\to\infty}\frac{x^2-1}{x^2}=\lim_{x\to\infty}\frac{\frac{x^2}{x^2}-\frac{1}{x^2}}{\frac{x^2}{x^2}}$$

$$=\lim_{x\to\infty}\frac{1-\frac{1}{x^2}}{1}$$

$$=\frac{1-0}{1}$$

$$=1$$

c)

$$\lim_{x\to\infty}\frac{x^3-1}{x^2}=\lim_{x\to\infty}\frac{\frac{x^3}{x^3}-\frac{1}{x^3}}{\frac{x^2}{x^3}}$$

$$=\lim_{x\to\infty}\frac{1-\frac{1}{x^3}}{\frac{1}{x}}$$

Since the numerator approaches 1 and the denominator approaches 0, the limit increases without bound.

- State the following facts about **limits at infinity for rational functions.** For the rational function $f(x) = N(x)/D(x)$, where $N(x) = a_n x^n + \cdots + a_0$ and $D(x) = b_m x^m + \cdots + b_0$, $\lim_{x\to\pm\infty} f(x) = \begin{cases} 0, & n < m \\ \frac{a_n}{b_m}, & n = m \end{cases}$.
  If $n > m$, then the limit does not exist.

Example 3. A company manufactures a product at a cost of \$2 per unit. Their fixed cost is \$1000 per week. What value does the average weekly cost per unit approach as the number of units increases without bound?

$$\lim_{x\to\infty}\frac{2x+1000}{x}=\lim_{x\to\infty}\left(2+\frac{1000}{x}\right)=\$2.$$

II. Limits of Sequences (pp. 887−888) Pace: 10 minutes

- State that if f is a function such that $\lim_{x\to\infty} f(x) = L$, and $\{a_n\}$ is a sequence such that $f(n) = a_n$ for all positive integers n, then $\lim_{x\to\infty} a_n = L$.

Example 4. Find the limit of the following sequences.

a) $\lim_{n\to\infty}\frac{3n}{n^2+1}=0$

b) $\lim_{n\to\infty}\frac{3n^2}{n^2+1}=3$

c) $\lim_{n\to\infty}\left[\frac{6}{n^4}\left(\frac{n^2(n+1)^2}{4}\right)\right]=\lim_{n\to\infty}\frac{6n^4+12n^3+6n^2}{4n^4}=\frac{3}{2}$