Chapter 12 Limits and an Introduction to Calculus

Course/Section
Lesson Number
Date

Section 12.5 The Area Problem

Section Objectives: Students will know how to find limits of summations and use them to find area of regions bounded by graphs of functions.

I. Limits of Summations (pp. 892–894) Pace: 15 minutes

- State that earlier we saw that $\sum_{i=1}^{\infty} a_1 r^{i-1} = \frac{a_1}{1-r}$. In developing this formula, what we actually did was find $\lim_{n\to\infty} \sum_{i=1}^{n} a_1 r^{i-1} = \lim_{n\to\infty} \frac{a_1(1-r^n)}{1-r} = \frac{a_1}{1-r}$.

- State the following **summation formulas and properties.**

1. $\sum_{i=1}^{n} c = cn$
2. $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
3. $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
4. $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$
5. $\sum_{i=1}^{n} (a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$
6. $\sum_{i=1}^{n} ka_i = k\sum_{i=1}^{n} a_i$

Example 1. Evaluate the following summations.

a)

$$\sum_{i=1}^{100} i^3 = \frac{100^2(100+1)^2}{4} = 25,502,500$$

b)

$$\sum_{i=1}^{n} \frac{2i-3}{n^2} = \frac{1}{n^2}\left(2\sum_{i=1}^{n} i - \sum_{i=1}^{n} 3\right)$$

$$= \frac{1}{n^2}\left(2\cdot\frac{n(n+1)}{2} - 3n\right)$$

$$= \frac{1}{n^2}\left(n^2 - 2n\right)$$

$$= \frac{n-2}{n}$$

Example 2. Evaluate the following limit.

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left(2+\frac{i}{n}\right)^2\frac{1}{n} = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\left(4+\frac{4i}{n}+\frac{i^2}{n^2}\right)$$
$$= \lim_{n\to\infty}\frac{1}{n}\left(\sum_{i=1}^{n}4+\frac{4}{n}\sum_{i=1}^{n}i+\frac{1}{n^2}\sum_{i=1}^{n}i^2\right)$$
$$= \lim_{n\to\infty}\frac{1}{n}\left(4n+\frac{4}{n}\cdot\frac{n(n+1)}{2}+\frac{1}{n^2}\cdot\frac{n(n+1)(2n+1)}{6}\right)$$
$$= \lim_{n\to\infty}\left(4+\frac{2n+2}{n}+\frac{2n^2+3n+1}{6n^2}\right)$$
$$= 4+2+\frac{2}{6}$$
$$= \frac{19}{3}$$

II. The Area Problem (pp. 895−897) Pace: 15 minutes

- State that we now find the area of a region bounded by a nonnegative, continuous function, the x-axis, the vertical line $x = a$, and the vertical line $x = b$. We do this by approximating the area of the region with rectangles, and then letting the number of rectangles approach infinity. First we will try just an approximation.

Example 3. Use four rectangles to approximate the area of the region bounded by $f(x) = 4 - x^2$, the x-axis, $x = 0$, and $x = 2$.

First divide the interval [0, 2] into four equal subintervals: [0, 0.5], [0.5, 1], [1, 1.5], [1.5, 2]. We can get the height of each rectangle by evaluating the function at the right endpoint of each subinterval. Note that the right endpoint of each subinterval is of the form $0.5i$. Hence the area of the region is approximately

$$\sum_{i=1}^{4}\left(4-\left(\frac{1}{2}i\right)^2\right)\frac{1}{2} = \frac{1}{2}\left(\sum_{i=1}^{4}4-\frac{1}{4}\sum_{i=1}^{4}i^2\right)$$
$$= \frac{1}{2}\left(4\cdot4-\frac{1}{4}\cdot\frac{4(4+1)(2\cdot4+1)}{6}\right)$$
$$= \frac{17}{4}$$

- State that the area of the region bounded by $y = f(x)$, $y = 0$, $x = a$, and $x = b$ is $A = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(a+\frac{(b-a)i}{n}\right)\left(\frac{b-a}{n}\right)$.

Example 4. Find the area of the region bounded by $f(x) = 4 - x^2$, the x-axis, $x = 0$, and $x = 2$.

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \left(4 - \left(0 + \frac{(2-0)i}{n}\right)^2\right)\left(\frac{2-0}{n}\right)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(4 - \frac{4i^2}{n^2}\right)\left(\frac{2}{n}\right)$$

$$= \lim_{n\to\infty} \frac{2}{n}\left(\sum_{i=1}^{n} 4 - \frac{4}{n^2}\sum_{i=1}^{n} i^2\right)$$

$$= \lim_{n\to\infty} \frac{2}{n}\left(4n - \frac{4}{n^2}\,\frac{n(n+1)(2n+1)}{6}\right)$$

$$= \lim_{n\to\infty}\left(8 - \frac{8n^2 + 12n + 4}{3n^2}\right)$$

$$= 8 - \frac{8}{3}$$

$$= \frac{16}{3}$$

Example 5. Find the area of the region bounded by the graphs of $f(x) = 1 + x^3$, $y = 0$, $x = 0$, and $x = 4$.

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \left(1 + \left(0 + \frac{(4-0)i}{n}\right)^3\right)\left(\frac{4-0}{n}\right)$$

$$= \lim_{n\to\infty} \frac{4}{n}\sum_{i=1}^{n} \left(1 + \left(\frac{4i}{n}\right)^3\right)$$

$$= \lim_{n\to\infty} \frac{4}{n}\left(\sum_{i=1}^{n} 1 + \frac{64}{n^3}\sum_{i=1}^{n} i^3\right)$$

$$= \lim_{n\to\infty} \frac{4}{n}\left(n + \frac{64}{n^3}\cdot\frac{n^2(n+1)^2}{4}\right)$$

$$= \lim_{n\to\infty}\left(4 + \frac{64n^2 + 128n + 64}{n^2}\right)$$

$$= 4 + 64$$

$$= 68$$