Chapter 2 Polynomial and Rational Functions

Course/Section
Lesson Number
Date

Section 2.5 Zeros of Polynomial Functions

Section Objectives: Students will know how to determine the number of rational and real zeros of polynomial functions, and how to find the zeros.

I. The Fundamental Theorem of Algebra (p. 169) Pace: 5 minutes

- State the **Fundamental Theorem of Algebra.**

  If $f(x)$ is a polynomial function of degree n, where $n > 0$, then f has at least one zero in the complex number system.

- State the **Linear Factorization Theorem.**

  If $f(x)$ is a polynomial function of degree n, where $n > 0$, then f has precisely n linear factors

  $f(x) = a_n(x - c_1)(x - c_2)\cdots(x - c_n)$

  where $c_1, c_2, \ldots, c_n$ are complex zeros.

Tip: It should be pointed out that the zeros in the above theorem may not be distinct.

II. The Rational Zero Test (pp. 170−172) Pace: 15 minutes

- Ask the students how they would solve $x^3 + 6x - 7 = 0$. Then ask them how they would solve the same equation if they knew that, if there were any, the rational zeros would have to be in the list ±1, ±2, ±3, ±6. Now state the **Rational Zero Test.**

  If the polynomial $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1 + a_0$ has integer coefficients with $a_n \neq 0$ and $a_0 \neq 0$, then any rational zero of f will be of the form p/q, where p is a factor of a_0 and q is a factor of a_n.

  Example 1. Find the zeros of $x^3 - 7x - 6 = 0$.

  p: ±1, ±2, ±3, ±6
  q: ±1
  p/q: ±1, ±2, ±3, ±6

  Use synthetic division to find a number from the list that is a solution.

  $$\begin{array}{r|rrrr} -1 & 1 & 0 & -7 & -6 \\ & & -1 & 1 & 6 \\ \hline & 1 & -1 & -6 & \boxed{0} \end{array}$$

  We now have $(x + 1)(x^2 - x - 6) = 0$. $x - 1 = 0 \Rightarrow x = 1$.
  $x^2 - x - 6 = (x - 3)(x + 2) = 0 \Rightarrow x = -2$ or $x = 3$.

  The zeros are $x = 1$, $x = -2$, and $x = 3$.

Example 2. Find all real zeros of $3x^3 - 20x^2 + 23x + 10$.

p: ±1, ±2, ±5, ±10

q: ±1, ±3

p/q: ±1, ±2, ±5, ±10, ±1/3, ±2/3, ±5/3, ±10/3

$$\begin{array}{r|rrrr} 2 & 3 & -20 & 23 & 10 \\ & & 6 & -28 & -10 \\ \hline & 3 & -14 & -5 & 0 \end{array}$$

One zero is 2. Two more zeros come from solving

$$3x^2 - 14x - 5 = 0$$

$$(3x+1)(x-5) = 0$$

$$x - 5 = 0 \Rightarrow x = 5$$

$$3x + 1 = 0 \Rightarrow x = -\frac{1}{3}$$

III. Conjugate Pairs (p. 173) Pace: 5 minutes

- Note that in Examples 1(c) and 1(d) of the text, the two complex zeros were conjugates. State that if f is a polynomial function with real coefficients, then whenever $a + bi$ is a zero of f, $a - bi$ is also a zero of f.

Example 3. Find a fourth-degree polynomial function with real coefficients that has 0, 1, and i as zeros.

Since i is a zero, $-i$ is also a zero.

$$f(x) = x(x-1)(x-i)(x+i) = x^4 - x^3 + x^2 - x$$

IV. Factoring a Polynomial (pp. 173−175) Pace: 5 minutes

- State that the Linear Factorization Theorem, together with the above statement regarding complex zeros and conjugate pairs, leads to the following statement regarding factoring a polynomial over the reals.

  Every polynomial of degree $n > 0$ with real coefficients can be written as the product of linear and quadratic factors with real coefficients, where the quadratic factors have no real zeros.

Example 4. Find all zeros of $f(x) = x^4 - 4x^3 + 12x^2 + 4x - 13$, given that $2 + 3i$ is a zero.

Because $2 + 3i$ is a zero, $2 - 3i$ is also a zero. This means that $x^2 - 4x + 13$ is a factor of $f(x)$.

$$\begin{array}{r} x^2 - 1 \\ x^2 - 4x + 13 \overline{\big) x^4 - 4x^3 + 12x^2 + 4x - 13} \\ \underline{x^4 - 4x^3 + 13x^2} \qquad\qquad \\ -x^2 + 4x - 13 \\ \underline{-x^2 + 4x - 13} \\ 0 \end{array}$$

All the zeros of f are -1, 1, $2 + 3i$, $2 - 3i$.

V. Other Tests for Zeros of Polynomials (pp. 176−178)
Pace: 10 minutes

- There are a couple of ways of dealing with a very large list generated by the Rational Zero Test. The first of these is **Descartes's Rule of Signs,** which states:

  Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial with real coefficients and $a_0 \neq 0$.

  1. The number of *positive real zeros* of f is either equal to the number of variations in sign of $f(x)$ or less than that number by an even integer.
  2. The number of *negative real zeros* of f is either equal to the number of variations in sign of $f(-x)$ or less than that number by an even integer.

  Two notes about Descartes's Rule of Signs:

  1. A variation in sign means that two consecutive coefficients have opposite signs.
  2. When we count the zeros, we must count their multiplicities.

  Example 5. Describe the possible real zeros of
  $f(x) = 7x^3 + 3x^2 - 5x + 9$.
  $f(x)$ has two variations in sign; therefore, there are either two or no positive real zeros.
  $f(-x) = -7x^3 + 3x^2 + 5x + 9$.
  $f(-x)$ has one variation in sign; therefore, there is exactly one negative real zero.

- The second way of dealing with a very large list generated by the Rational Zero Test is the **Upper and Lower Bound Rules**. Before you state this rule, discuss what upper and lower bounds are. A real number b is an **upper bound** for the real zeros of f if there are no zeros of f greater than b. A real number b is a **lower bound** for the real zeros of f if there are no zeros of f less than b.
- **Upper and Lower Bound Rules**
  Let $f(x)$ be a polynomial function with real coefficients and a positive leading coefficient. Suppose $f(x)$ is divided by $x - c$ using synthetic division.
  1. If $c > 0$ and each number in the last row is either positive or zero, then c is an *upper bound* for the real zeros of f.
  2. If $c < 0$ and the numbers in the last row are alternately positive and negative (zero entries count as either positive or negative), then c is a *lower bound* for the real zeros of f.

Example 6. Find all real zeros of $f(x) = x^4 - 3x^3 + x - 3$.

p: ±1, ±3

q: ±1

p/q: ±1, ±3

$f(x)$ has three variations in sign; therefore $f(x)$ has 3 or 1 positive real zeros.

$f(-x)$ has one variation in sign; therefore $f(x)$ has exactly 1 negative real zero. We will start by trying to find it.

$$\begin{array}{r|rrrrr} -1 & 1 & -3 & 0 & 1 & -3 \\ & & -1 & 4 & -4 & 3 \\ \hline & 1 & -4 & 4 & -3 & \boxed{0} \end{array}$$

Now we look for the positive zero. Testing the value 1 does not work, so 3 must be a zero.

$$\begin{array}{r|rrrr} 3 & 1 & -4 & 4 & -3 \\ & & 3 & -3 & 3 \\ \hline & 1 & -1 & 1 & \boxed{0} \end{array}$$

Now we solve $x^2 - x + 1 = 0$ by using the quadratic formula.

$$\begin{aligned} x &= \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(1)}}{2(1)} \\ &= \frac{1 \pm \sqrt{-3}}{2} \\ &= \frac{1}{2} \pm \frac{\sqrt{3}}{2} i \end{aligned}$$

Therefore, all four zeros of f are –1, 3, and $\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$.