Chapter 5 Analytic Trigonometry

Course/Section
Lesson Number
Date

Section 5.5 Multiple-Angle and Product-to-Sum Formulas

Section Objectives: Students will know how to use multiple-angle formulas, power-reducing formulas, half-angle formulas, and product-to-sum formulas to rewrite and evaluate trigonometric functions.

I. Multiple-Angle Formulas (pp. 407−409) Pace: 15 minutes

- State the following **double-angle formulas**.

$\sin 2u = 2\sin u \cos u$

$\cos 2u = \cos^2 u - \sin^2 u$

$= 2\cos^2 u - 1$

$= 1 - 2\sin^2 u$

$$\tan 2u = \frac{2\tan u}{1 - \tan^2 u}$$

Example 1. Solve $\sin 2x - \cos x = 0$.

$\sin 2x - \cos x = 0$

$2\sin x \cos x - \cos x = 0$

$\cos x\,(2\sin x - 1) = 0$

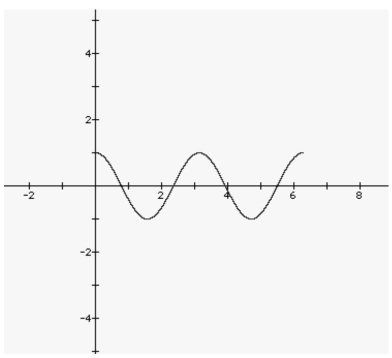
$\cos x = 0$ or $\sin x = 1/2$

$x = \pi/2, 3\pi/2$ or $x = \pi/6, 5\pi/6$

Example 2. Sketch the graph of $y = \cos^4 x - \sin^4 x$ on $[0, 2\pi]$.

$y = \cos^4 x - \sin^4 x = (\cos^2 x - \sin^2 x)(\cos^2 x + \sin^2 x)$

$= \cos 2x \bullet 1 = \cos 2x$



Example 3. Given $\sin x = 12/13$ and $\pi/2 < x < \pi$, find $\sin 2x$, $\cos 2x$, and $\tan 2x$.

$\cos x = -5/13$

$\sin 2x = 2\sin x \cos x = 2(12/13)(-5/13) = -120/169$

$\cos 2x = 1 - 2\sin^2 x = 1 - 2(12/13)^2 = -119/169$

$$\tan 2x = \frac{\sin 2x}{\cos 2x} = \frac{120}{119}$$

II. Power-Reducing Formulas (p. 409) Pace: 5 minutes

- State the following **power-reducing formulas**.

$$\sin^2 u = \frac{1-\cos 2u}{2} \quad \cos^2 u = \frac{1+\cos 2u}{2} \quad \tan^2 u = \frac{1-\cos 2u}{1+\cos 2u}$$

Example 4. Rewrite $\cos^4 x$ as a sum of first powers of the cosine function.

$$\begin{aligned}\cos^4 x &= \left(\cos^2 x\right)^2\\ &= \left(\frac{1+\cos 2x}{2}\right)^2\\ &= \frac{1}{4}\left(1+2\cos 2x+\cos^2 2x\right)\\ &= \frac{1}{4}\left(1+2\cos 2x+\frac{1+\cos 4x}{2}\right)\\ &= \frac{1}{8}\left(3+4\cos 2x+\cos 4x\right)\end{aligned}$$

III. Half-Angle Formulas (pp. 410−411) Pace: 10 minutes

- State the following **half-angle formulas.**

$$\sin\frac{u}{2} = \pm\sqrt{\frac{1-\cos u}{2}}$$

$$\cos\frac{u}{2} = \pm\sqrt{\frac{1+\cos u}{2}}$$

$$\tan\frac{u}{2} = \frac{1-\cos u}{\sin u} = \frac{\sin u}{1+\cos u}$$

Example 5. Find the exact value of cos 165°.

$$\cos 165^\circ = -\sqrt{\frac{1+\cos 330^\circ}{2}} = -\sqrt{\frac{1+\sqrt{3}/2}{2}} = -\frac{\sqrt{2+\sqrt{3}}}{2}$$

Example 6. Solve $2\sin^2 (x/2) = \cos x$ on $[0, 2\pi)$.

$$\begin{aligned}2\sin^2\frac{x}{2} &= \cos x\\ 2\left(\pm\sqrt{\frac{1-\cos x}{2}}\right)^2 &= \cos x\\ 2\left(\frac{1-\cos x}{2}\right) &= \cos x\\ 1-\cos x &= \cos x\\ 1 &= 2\cos x\\ \cos x &= \frac{1}{2}\\ x &= \frac{\pi}{3},\ \frac{5\pi}{3}\end{aligned}$$

IV. **Product-to-Sum Formulas** (pp. 411–413) Pace: 15 minutes

- State the following **product-to-sum formulas.**

$$\sin u \sin v = \frac{1}{2}\left[\cos(u-v) - \cos(u+v)\right]$$

$$\cos u \cos v = \frac{1}{2}\left[\cos(u-v) + \cos(u+v)\right]$$

$$\sin u \cos v = \frac{1}{2}\left[\sin(u+v) + \sin(u-v)\right]$$

$$\cos u \sin v = \frac{1}{2}\left[\sin(u+v) - \sin(u-v)\right]$$

Example 7. Rewrite sin 2*x* sin *x* as a difference.

$$\sin 2x \sin x = \frac{1}{2}\left[\cos(2x-x) - \cos(2x+x)\right] = \frac{\cos x}{2} - \frac{\cos 3x}{2}$$

- State the following **sum-to-product formulas.**

$$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

$$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

Example 8. Find the exact value of sin 195° − sin 105°.

$$\sin 195^\circ - \sin 105^\circ = 2\cos\left(\frac{195^\circ + 105^\circ}{2}\right)\sin\left(\frac{195^\circ - 105^\circ}{2}\right)$$

$$= 2\cos 150^\circ \sin 45^\circ = 2\left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = -\frac{\sqrt{6}}{2}$$

Example 9. Solve cos 3*x* + cos *x* = 0 on [0, 2π).

$$\cos 3x + \cos x = 0$$

$$2\cos\left(\frac{3x+x}{2}\right)\cos\left(\frac{3x-x}{2}\right) = 0$$

$$2\cos 2x \cos x = 0$$

$$\cos 2x = 0$$

$$2x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$$

$$x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

$$\cos x = 0 \Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}$$

Example 10. Verify the identity.

$$\frac{\sin 7x - \sin 5x}{\cos 7x - \cos 5x} = -\cot 6x$$

$$\frac{\sin 7x - \sin 5x}{\cos 7x - \cos 5x} = \frac{2\cos\left(\frac{7x+5x}{2}\right)\sin\left(\frac{7x-5x}{2}\right)}{-2\sin\left(\frac{7x+5x}{2}\right)\sin\left(\frac{7x-5x}{2}\right)}$$

$$= -\frac{\cos 6x}{\sin 6x}$$

$$= -\cot 6x$$

V. Application (p. 414) Pace: 10 minutes

- Assign the *Writing About Mathematics* on page 414 of the text.