Chapter 6 Additional Topics in Trigonometry

Course/Section
Lesson Number
Date

Section 6.3 Vectors in the Plane

Section Objectives: Students will know how to write the component forms of vectors and perform basic vector operations, how to write vectors as linear combinations of unit vectors, and how to find the direction angles of vectors.

I. Introduction (p. 447) Pace: 5 minutes

- State that some quantities have both a magnitude and a direction and therefore cannot be represented by a single real number. Such quantities can be represented by a **directed line segment**. A directed line segment has an **initial point** and a **terminal point**. Its **magnitude**, denoted by $\|\mathbf{v}\|$, can be found by using the Distance Formula.

- State that two vectors are equivalent if and only if they have the same direction and magnitude. The set of all directed line segments that are equivalent to a given line segment is called a **vector v in the plane**.

Example 1. Let **v** be the vector from (0, 0) to (4, 5) and **u** be the vector from (2, 3) to (6, 8). Show that **v** = **u**.

$$\|\mathbf{v}\| = \sqrt{(4-0)^2 + (5-0)^2} = \sqrt{41}$$

$$\|\mathbf{u}\| = \sqrt{(6-2)^2 + (8-3)^2} = \sqrt{41}$$

Since both vectors have the same slope and the same magnitude, they are equal.

II. Component Form of a Vector (p. 448) Pace: 5 minutes

- State that a vector is in **standard position** if its initial point is at the origin. Also, if **v** is in standard position and its terminal point is (v_1, v_2), then **v** can be uniquely represented as $\mathbf{v} = \langle v_1, v_2 \rangle$. This is called the **component form of v**. Furthermore, if **v** is the vector from (x_1, y_1) to (x_2, y_2), then its component form is $\mathbf{v} = \langle x_2 - x_1, y_2 - y_1 \rangle$.

- State that the **magnitude** of $\mathbf{v} = \langle v_1, v_2 \rangle$ is $\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2}$. A **unit vector** has magnitude 1. The zero vector $\mathbf{0} = \langle 0, 0 \rangle$ has magnitude zero.

Example 2. Find the component form and the magnitude of the vector from (2, −4) to (5, 7).

$\mathbf{v} = \langle 5 - 2, 7 - (-4) \rangle = \langle 3, 11 \rangle$.

$$\|\mathbf{v}\| = \sqrt{3^2 + 11^2} = \sqrt{130}$$

III. Vector Operations (pp. 449−451) Pace: 15 minutes

- State the following definitions of vector operations. Let $\mathbf{u} = \langle u_1, u_2\rangle$ and $\mathbf{v} = \langle v_1, v_2\rangle$ be vectors and k a real number, called a **scalar.** Then $\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2\rangle$ and $k\mathbf{u} = k\langle u_1, u_2\rangle = \langle ku_1, ku_2\rangle$.

- State that geometrically, $k\mathbf{u}$ is a vector that is $|k|$ times as long as **u**, in the same direction if $k > 0$ and in the opposite direction if $k < 0$.

- State that geometrically, **u** + **v** is the vector formed by the initial point of **u** and the terminal point of **v**, when the initial point of **v** is placed at the terminal point of **u**.

- State that if $\mathbf{v} = \langle v_1, v_2\rangle$, then $-\mathbf{v} = \langle -v_1, -v_2\rangle$ and $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v}) = \langle u_1 - v_1, u_2 - v_2\rangle$.

  Example 3. Let $\mathbf{u} = \langle -5, 2\rangle$ and $\mathbf{v} = \langle 6, -3\rangle$. Find the following.

  a) $4\mathbf{u} = 4\langle -5, 2\rangle = \langle -20, 8\rangle$

  b) $\mathbf{u} + \mathbf{v} = \langle -5 + 6, 2 + (-3)\rangle = \langle 1, -1\rangle$

  c) $2\mathbf{u} - \mathbf{v} = 2\langle -5, 2\rangle - \langle 6, -3\rangle = \langle -10, 4\rangle - \langle 6, -3\rangle = \langle -16, 7\rangle$

- Discuss the properties on page 451 of the text.

IV. Unit Vectors (pp. 451−452) Pace: 10 minutes

- State that if $\mathbf{v} \neq \mathbf{0}$, then the unit vector in the direction of **v** is

$$\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v}.$$

  Example 4. Find the unit vector in the direction of $\mathbf{v} = \langle 3, -4\rangle$.

  $\|\mathbf{v}\| = \sqrt{3^2 + (-4)^2} = 5$

  $\mathbf{u} = \frac{1}{5}\langle 3, -4\rangle = \left\langle \frac{3}{5}, \frac{-4}{5}\right\rangle$

- Define the standard unit vectors to be $\mathbf{i} = \langle 1, 0\rangle$ and $\mathbf{j} = \langle 0, 1\rangle$. Note that $\mathbf{v} = \langle v_1, v_2\rangle = \langle v_1, 0\rangle + \langle 0, v_2\rangle = v_1\langle 1, 0\rangle + v_2\langle 0, 1\rangle = v_1\mathbf{i} + v_2\mathbf{j}$, where v_1 and v_2 are the horizontal and vertical components of **v**, and $v_1\mathbf{i} + v_2\mathbf{j}$ is a linear combination of the vectors **i** and **j**.

  Example 5. Express $\mathbf{v} = \langle 3, -4\rangle$ as a linear combination of the vectors **i** and **j**.

  $\mathbf{v} = \langle 3, -4\rangle = 3\mathbf{i} - 4\mathbf{j}$

V. Direction Angles (p. 453) Pace: 10 minutes

- State that if $\mathbf{u} = \langle x, y \rangle$ is a unit vector in standard position and θ is the angle in standard position with terminal side at **u**, then
  $\mathbf{u} = \langle x, y \rangle = \langle \cos\theta, \sin\theta \rangle = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j}$.
  The angle θ is called the **direction angle** of the vector **u**.

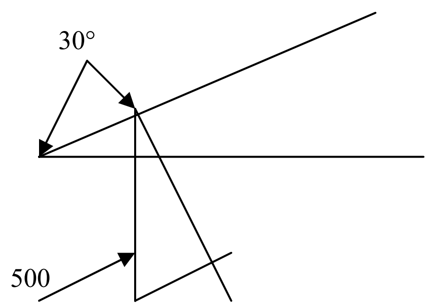
- State that, in general, if $\mathbf{v} = \langle a, b \rangle$ is a vector in standard position and θ is the angle in standard position with terminal side at **v**, then
  $\mathbf{v} = \langle a, b \rangle = \|\mathbf{v}\|\langle \cos\theta, \sin\theta \rangle = \|\mathbf{v}\|(\cos\theta)\mathbf{i} + \|\mathbf{v}\|(\sin\theta)\mathbf{j}$. Also, $\tan\theta = b/a$.

Example 6. Find the direction angle for $\mathbf{v} = 2\mathbf{i} - \mathbf{j}$.
$\tan\theta = -1/2$ and $(2, -1)$ is in the fourth quadrant, so
$\theta = 360° + \tan^{-1}(-1/2) \approx 333.4°$.

VI. Applications of Vectors (pp. 454−455) Pace: 15 minutes

Example 7. A piano weighing 500 lb is being pushed up a ramp into the back of a truck. The ramp is a board that can support 450 lb and makes a 30° angle with the horizontal. Will the ramp support the piano?

We need to find the magnitude of the force perpendicular to the board.



$\|\mathbf{v}\| = 500 \cos 30° \approx 433$ lb. So the board will support the piano.

Example 8. A plane has set a bearing of S 60° E at 600 mph, but there is a wind blowing in the direction of N 45° E at 80 mph. What is the resultant speed and direction of the plane?

Plane: $\mathbf{p} = 600\cos 330°\mathbf{i} + 600\sin 330°\mathbf{j} = 300\sqrt{3}\,\mathbf{i} - 300\mathbf{j}$

Wind: $\mathbf{w} = 80\cos 45° + 80\sin 45° = 40\sqrt{2}\,\mathbf{i} + 40\sqrt{2}\,\mathbf{j}$

Resultant: $\mathbf{p} + \mathbf{w} = (300\sqrt{3}\,\mathbf{i} - 300\mathbf{j}) + (40\sqrt{2}\,\mathbf{i} + 40\sqrt{2}\,\mathbf{j})$

$= (300\sqrt{3} + 40\sqrt{2})\mathbf{i} + (-300 + 40\sqrt{2})\mathbf{j}$

$\approx 576.18\mathbf{i} - 356.57\mathbf{j}$

The speed is $\|\mathbf{p} + \mathbf{w}\| \approx \sqrt{576.18^2 + (-356.57)^2} \approx 677.59$ mph.

The direction is $360° + \tan^{-1}(-356.57/576.18) \approx 328.2°$, or S 58.2° E.