Chapter 6 Additional Topics in Trigonometry

Course/Section Lesson Number Date

Section 6.4 Vectors and Dot Products

Section Objectives: Students will know how to find the dot product of two vectors and the angle between two vectors, how to determine whether two vectors are orthogonal, and how to write a vector as the sum of two vector components.

I. The Dot Product of Two Vectors (pp. 460−461)   Pace: 10 minutes

- State that the **dot product** of the vectors $\mathbf{u} = \langle u_1, u_2\rangle$ and $\mathbf{v} = \langle v_1, v_2\rangle$ is $\mathbf{u} \bullet \mathbf{v} = u_1 v_1 + u_2 v_2$.

- Discuss the properties on page 460 of the text.

Example 1. Let $\mathbf{u} = \langle 2, 6\rangle$, $\mathbf{v} = \langle -1, 5\rangle$, and $\mathbf{w} = \langle -3, 1\rangle$. Find the following.

a) $\mathbf{u} \bullet \mathbf{v} = \langle 2, 6\rangle \bullet \langle -1, 5\rangle = 2(-1) + 6(5) = 28$

b) $\mathbf{u} \bullet (\mathbf{v} + \mathbf{w}) = \mathbf{u} \bullet \mathbf{v} + \mathbf{u} \bullet \mathbf{w} = 28 + 0 = 28$

II. The Angle Between Two Vectors (pp. 461−463)   Pace: 10 minutes

- State that the **angle between two nonzero vectors** is the angle θ, $0 \le \theta \le \pi$, between their respective standard position vectors.

- State that the angle θ between two nonzero vectors **u** and **v** can be found by using $\cos\theta = \dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$.

Example 2. Find the angle between $\mathbf{u} = \langle -2, 3\rangle$ and $\mathbf{v} = \langle 1, -5\rangle$.

$$\theta = \cos^{-1}\frac{\langle -2,3\rangle\cdot\langle 1,-5\rangle}{\|\langle -2,3\rangle\|\|\langle 1,-5\rangle\|} = \cos^{-1}\left(-\frac{17}{5\sqrt{26}}\right) \approx 131.8^\circ$$

- State that the above formula can be rewritten to produce an alternative form of the dot product, $\mathbf{u} \bullet \mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\| \cos\theta$.

- State that because cos 90° = 0, we can say that two vectors **u** and **v** are **orthogonal** if $\mathbf{u} \bullet \mathbf{v} = 0$. Note that this implies that the zero vector is orthogonal to every vector.

Example 3. Are the two vectors $\mathbf{u} = \langle -4, 2\rangle$ and $\mathbf{v} = \langle 1, 2\rangle$ orthogonal?
$\mathbf{u} \bullet \mathbf{v} = -4(1) + 2(2) = 0$. So, **u** and **v** are orthogonal.

III. Finding Vector Components (pp. 463−465)   Pace: 15 minutes

- State that we now do the reverse of adding two vectors to find their resultant: we will decompose a vector into the sum of **vector components**.

- State that if **u** is a nonzero vector such that $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$, where $\mathbf{w}_1$ and $\mathbf{w}_2$ are orthogonal, then $\mathbf{w}_1$ and $\mathbf{w}_2$ are called vector components of **u.**

- State that the **projection of u onto v**, where **u** and **v** are nonzero, is $\text{proj}_{\mathbf{v}}\mathbf{u} = \left(\dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v}$. State that in the above definition of vector components, $\mathbf{w}_1 = \text{proj}_{\mathbf{v}}\mathbf{u}$ for some nonzero vector **v**, and $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1$.

Example 4. Find the projection of $\mathbf{u} = \langle 5, 2\rangle$ onto $\mathbf{v} = \langle 3, -1\rangle$. Then write $\mathbf{u}$ as the sum of two orthogonal vectors, one of which is $\text{proj}_{\mathbf{v}}\mathbf{u}$.

$$\mathbf{w}_1 = \text{proj}_{\mathbf{v}}\mathbf{u} = \frac{13}{10}\langle 3, -1\rangle = \left\langle \frac{39}{10}, -\frac{13}{10} \right\rangle$$

$$\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = \langle 5, 2\rangle - \left\langle \frac{39}{10}, -\frac{13}{10} \right\rangle = \left\langle \frac{11}{10}, \frac{33}{10} \right\rangle$$

$$\mathbf{u} = \left\langle \frac{39}{10}, -\frac{13}{10} \right\rangle + \left\langle \frac{11}{10}, \frac{33}{10} \right\rangle$$

Example 5. A 500-pound piano sits on a ramp that is inclined at 45°. What force is required to keep the piano from rolling down the ramp? We need the projection of $\mathbf{F} = -500\mathbf{j}$ onto $\mathbf{v} = \cos 45°\mathbf{i} + \sin 45°\mathbf{j}$.

$$\text{proj}_{\mathbf{v}}\mathbf{F} = \left(\frac{-250\sqrt{2}}{1}\right)\left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle = -250\sqrt{2}\left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$$

So, the force required to keep the piano from rolling down the ramp is $250\sqrt{2} \approx 353.6$.

IV. Work (p. 466) Pace: 5 minutes

- State that the **work** W done by a constant force $\mathbf{F}$ as its point of application moves along the vector $\overrightarrow{PQ}$ is given by either of the following.
  - $W = \|\text{proj}_{\overrightarrow{PQ}}\mathbf{F}\| \|\overrightarrow{PQ}\|$
  - $W = \mathbf{F} \bullet \overrightarrow{PQ}$

Example 6. A man pushes a broom with a constant force of 40 pounds. The handle of the broom is at an angle of 30°. How much work is done pushing the broom 30 feet?
$W = \mathbf{F} \bullet 30\mathbf{i} = 40(\cos 30°\mathbf{i} + \sin 30°\mathbf{j}) \bullet 30\mathbf{i} = 120\cos 30° \approx 103.9$ foot-pounds.