Chapter 6 Additional Topics in Trigonometry

Course/Section
Lesson Number
Date

Section 6.5 Trigonometric Form of a Complex Number

Section Objectives: Students will know how to multiply and divide complex numbers in trigonometric form and find powers and *n*th roots of complex numbers.

I. The Complex Plane (p. 470) Pace: 5 minutes

- State that plotting a complex number $z = a + bi$ in the **complex plane** is the same as plotting the point (a, b) in the rectangular coordinate system. Also, state that in the complex plane, the horizontal axis is called the **real axis** and the vertical axis is called the **imaginary axis**.

- State that the **absolute value** of a complex number $z = a + bi$ is the distance between (0, 0) and (a, b). Hence $|z| = |a + bi| = \sqrt{a^2 + b^2}$.

  Example 1. Find the absolute value of $z = 4 - 3i$.
  $|z| = |4 - 3i| = \sqrt{4^2 + (-3)^2} = 5$

II. Trigonometric Form of a Complex Number (pp. 471−472)
Pace: 10 minutes

- State that if θ is an angle in standard position with terminal side containing the point (a, b) and $r = |a + bi|$, then $a = r\cos\theta$ and $b = r\sin\theta$. Thus $z = a + bi = (r\cos\theta) + (r\sin\theta)i = r(\cos\theta + i\sin\theta)$. This is called the **trigonometric form of a complex number**. Furthermore, state that r is called the **modulus** of z, θ is called the **argument** of z, and $\tan\theta = b/a$.

  Example 2. Write $z = -2 + 2i$ in trigonometric form.
  $\tan\theta = 2/(-2) = -1$ and z is in the second quadrant; therefore, $\theta = 3\pi/4$.
  $r = \sqrt{(-2)^2 + 2^2} = 2\sqrt{2}$ and $z = 2\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

  Example 3. Write $4(\cos 5\pi/6 + i\sin 5\pi/6)$ in $a + bi$ form.
  $4(\cos 5\pi/6 + i\sin 5\pi/6) = 4\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = -2\sqrt{3} + 2i$.

III. Multiplication and Division of Complex Numbers (pp. 472−473)
Pace: 10 minutes

- State that if $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then
  $z_1z_2 = [r_1(\cos\theta_1 + i\sin\theta_1)][r_2(\cos\theta_2 + i\sin\theta_2)]$
  $= r_1r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_2\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)]$
  $= r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$
  Similarly, $z_1/z_2 = (r_1/r_2)[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$.

  Example 4. If $z_1 = 8(\cos 120° + i\sin 120°)$ and $z_2 = 6(\cos 150° + i\sin 150°)$, find the following.
  a) $z_1z_2 = 48[\cos(120° + 150°) + i\sin(120° + 150°)]$
  $= 48[\cos(270°) + i\sin(270°)] = -48i$

  b) $z_1/z_2 = (8/6)[\cos(120° - 150°) + i\sin(120° - 150°)]$
  $= (4/3)[\cos(-30°) + i\sin(-30°)] = \frac{2\sqrt{3}}{3} - \frac{2}{3}i$

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IV. Powers of Complex Numbers (p. 474) Pace: 10 minutes

- State that the product formula can be generalized to any finite number of factors $[r_1(\cos\theta_1 + i\sin\theta_1)]\,[r_2(\cos\theta_2 + i\sin\theta_2)]\cdots[r_n(\cos\theta_n + i\sin\theta_n)]$ $= (r_1r_2\cdots r_n)[\cos(\theta_1 + \theta_2 + \cdots + \theta_n) + i\sin(\theta_1 + \theta_2 + \cdots + \theta_n)]$.

- State that if, in the above formula, all the moduli were the same and all the arguments were the same, we would get **DeMoivre's Theorem.**
$z^n = [r(\cos\theta + i\sin\theta)]^n = r^n[\cos(n\theta) + i\sin(n\theta)]$.

Example 5. Evaluate $\left(-2\sqrt{3} - 2i\right)^5$.

$$\left(-2\sqrt{3} - 2i\right)^5 = \left[4\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right)\right]^5$$
$$= 4^5\left(\cos 5\left(\frac{7\pi}{6}\right) + i\sin 5\left(\frac{7\pi}{6}\right)\right)$$
$$= 1024\left(\cos\left(\frac{35\pi}{6}\right) + i\sin\left(\frac{35\pi}{6}\right)\right)$$
$$= 512\sqrt{3} - 512i$$

V. Roots of Complex Numbers (pp. 475–477) Pace: 15 minutes

- State that if $u = s(\cos\beta + i\sin\beta)$ is an nth root of $z = r(\cos\theta + i\sin\theta)$, then $[s(\cos\beta + i\sin\beta)]^n = s^n(\cos n\beta + i\sin n\beta) = r(\cos\theta + i\sin\theta)$. This yields $s^n = r$ and $n\beta = \theta + 2\pi k$, where $k = 0, 1, 2, \ldots, n - 1$, or
$s = \sqrt[n]{r}$ and $\beta = \dfrac{\theta + 2\pi k}{n}$.

- State that the ***n*th root** of $z = r(\cos\theta + i\sin\theta)$ is
$\sqrt[n]{r}\left(\cos\dfrac{\theta + 2\pi k}{n} + i\sin\dfrac{\theta + 2\pi k}{n}\right)$, where $k = 0, 1, 2, \ldots, n - 1$.

Example 6. Find the four fourth roots of 16.
$16 = 16(\cos 0 + i\sin 0)$. So, the fourth roots are

$$\sqrt[4]{16}\left(\cos\frac{0 + 2\pi\cdot 0}{4} + i\sin\frac{0 + 2\pi\cdot 0}{4}\right) = 2$$
$$\sqrt[4]{16}\left(\cos\frac{0 + 2\pi\cdot 1}{4} + i\sin\frac{0 + 2\pi\cdot 1}{4}\right) = 2i$$
$$\sqrt[4]{16}\left(\cos\frac{0 + 2\pi\cdot 2}{4} + i\sin\frac{0 + 2\pi\cdot 2}{4}\right) = -2$$
$$\sqrt[4]{16}\left(\cos\frac{0 + 2\pi\cdot 3}{4} + i\sin\frac{0 + 2\pi\cdot 3}{4}\right) = -2i$$

Example 7. Find the three third roots of $4\sqrt{3} - 4i$.

$$2\left(\cos\frac{11\pi/6 + 2\pi\cdot 0}{3} + i\sin\frac{11\pi/6 + 2\pi\cdot 0}{3}\right) \approx -0.6840 + 1.8794i$$
$$2\left(\cos\frac{11\pi/6 + 2\pi\cdot 1}{3} + i\sin\frac{11\pi/6 + 2\pi\cdot 1}{3}\right) \approx -1.2856 - 1.5321i$$
$$2\left(\cos\frac{11\pi/6 + 2\pi\cdot 2}{3} + i\sin\frac{11\pi/6 + 2\pi\cdot 2}{3}\right) \approx 1.9696 - 0.3473i$$

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