Chapter 8 Matrices and Determinants

Course/Section
Lesson Number
Date

Section 8.1 Matrices and Systems of Equations

Section Objectives: Students will know how to perform elementary row operations on matrices and how to use matrices, Gaussian elimination, and Gauss-Jordan elimination to solve systems of linear equations.

I. Matrices (pp. 572−573) Pace: 10 minutes

- State that a matrix is a rectangular array of real numbers. We will use the double subscript notation--that is, the element of matrix A in the ith row and the jth column is denoted by a_{ij}. The dimensions of a matrix are given as *rows* × *columns.* A matrix is square if it is $n \times n$; we say it has order n. The main diagonal of a square matrix is all the elements a_{ij} for which $i = j$.
- Relate matrices to systems of equations by saying that we are going to write a system of equations without the variables, addition signs, and equal signs. So, if the system of equations is

$$\begin{cases} a_{11}x + a_{12}y + a_{13}z = b_1 \\ a_{21}x + a_{22}y + a_{23}z = b_2 \\ a_{31}x + a_{32}y + a_{33}z = b_3 \end{cases},$$

then the augmented matrix is

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix},$$

and the coefficient matrix is

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

Note that any time a term is missing from a system of equations, we must put a zero in its place in the matrix.

II. Elementary Row Operations (pp. 574−576) Pace: 15 minutes

- Translate the **elementary row operations** from Section 7.3 into the language of matrices. Note that these operations will transform the augmented matrix of a system of equations into the augmented matrix of an equivalent system of equations. If one matrix has been obtained from another matrix by using elementary row operations, then the two matrices are said to **row-equivalent**. Here are the operations.
  1. Interchange two rows.
  2. Replace any row by a nonzero multiple of itself.
  3. Replace any row by the sum of itself and a multiple of any other row in the matrix.

Tip: It should be emphasized that the verb is *replace*. It is not add or multiply; it is *replace*.

- Discuss the *Technology* on page 574 of the text.

Example 1. Use an augmented matrix to solve the system of equations.

$$\begin{cases} x + \ y + 2z = 3 \\ 3x + 4y + 4z = 9 \\ 5x + 2y + 15z = 13 \end{cases}$$

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 3 & 4 & 4 & 9 \\ 5 & 2 & 15 & 13 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & -2 & 0 \\ 0 & -3 & 5 & -2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & -1 & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

The solution is (−5, 4, 2).

- State that this last matrix is in **row-echelon form.** Here are the characteristics that define a matrix in row-echelon form.
  1. All rows consisting entirely of zeros occur at the bottom of the matrix.
  2. The first nonzero element of any row is 1, called a leading 1.
  3. For two successive nonzero rows, the leading 1 in the upper row is farther to the left than the leading 1 in the lower row.
  - State that if we add one more condition, we will have the definition of **reduced row-echelon form.**
  4. Every column with a leading 1 has zeros everywhere else in that column.

Example 2. Which of the following matrices are in reduced row-echelon form? If it is not in reduced row-echelon form, state whether it is in row-echelon form.

a) $\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 4 \end{bmatrix}$

reduced row-echelon form

b) $\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 8 \end{bmatrix}$

reduced row-echelon form

c) $\begin{bmatrix} 1 & 5 & 4 \\ 0 & 1 & 2 \end{bmatrix}$

just row-echelon form

d) $\begin{bmatrix} 1 & 0 & 2 & 5 \\ 0 & 1 & 3 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

reduced row-echelon form

III. Gaussian Elimination with Back-Substitution (pp. 577−578)
Pace: 5 minutes

- State that solving a system of equations by **Gaussian elimination** is taking the augmented matrix of the system of equations and transforming it into row-echelon form, and then using back-substitution to find the values of the variables.

Example 3. Solve by Gaussian elimination.

$$\begin{cases} x + 2y + 3z = 4 \\ 3x + 7y + 11z = 15 \\ -2x - 2y - z = 0 \end{cases}$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 7 & 11 & 15 \\ -2 & -2 & -1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 5 & 8 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

The solution is (0, −1, 2).

IV. Gauss-Jordan Elimination (pp. 579−581) Pace: 15 minutes

- State that solving a system of equations by **Gauss-Jordan elimination** is taking the augmented matrix of the system of equations and transforming it into reduced row-echelon form. The last column gives the values of the variables.

Tip: Work column by column. Get the first column the way you want it, then move to the second column, and so on. Students will at first work in a seemingly random order and hence have many counterproductive steps.

Example 4. Solve by Gauss-Jordan elimination.

a) $\begin{cases} 2x + 4y = 6 \\ 3x + 7y = 5 \end{cases}$

$$\begin{bmatrix} 2 & 4 & 6 \\ 3 & 7 & 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 3 & 7 & 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -4 \end{bmatrix}$$

The solution is (5, −4).

b) $\begin{cases} x + 2y + 3z = 1 \\ x + 3y + z = -1 \\ 3x + 7y + 7z = 6 \end{cases}$

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 1 & -1 \\ 3 & 7 & 7 & 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & -2 & -2 \\ 0 & 1 & -2 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

There is no solution.

c) $\begin{cases} x + 2y - 3z = 2 \\ 2x + 5y - 6z = 1 \end{cases}$

$$\begin{bmatrix} 1 & 2 & -3 & 2 \\ 2 & 5 & -6 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & -3 & 2 \\ 0 & 1 & 0 & -3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -3 & 8 \\ 0 & 1 & 0 & -3 \end{bmatrix}$$

The solution is $(3a + 8, -3, a)$, where a is a real number.