Chapter 8 Matrices and Determinants

Section 8.5 Applications of Matrices and Determinants

Course/Section
Lesson Number
Date

Section Objectives: Students will know how to use Cramer's Rule to solve systems of linear equations, and how to use determinants and matrices to model and solve problems.

I. Cramer's Rule (pp. 619−621) Pace: 20 minutes

- State that if we solve

  $$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$$

  by elimination, we obtain

  $$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1} = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{D_x}{D}$$

  and

  $$y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1} = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{D_y}{D}$$

Tip: Try to get the students to see the pattern. State that all denominators are equal to the determinant of the coefficient matrix. The numerator of x is the same as the denominator except that the x-coefficients are replaced by the constants; the numerator of y is the same as the denominator except that the y-coefficients are replaced by the constants.

Example 1. Use Cramer's Rule to solve the system of linear equations.

$$\begin{cases} 2x + 3y = 4 \\ 6x - 5y = 1 \end{cases}$$

$$D = \begin{vmatrix} 2 & 3 \\ 6 & -5 \end{vmatrix} = -28$$

$$D_x = \begin{vmatrix} 4 & 3 \\ 1 & -5 \end{vmatrix} = -23 \qquad D_y = \begin{vmatrix} 2 & 4 \\ 6 & 1 \end{vmatrix} = -22$$

$$x = \frac{D_x}{D} = \frac{-23}{-28} = \frac{23}{28} \qquad y = \frac{D_y}{D} = \frac{-22}{-28} = \frac{11}{14}$$

- Generalize Cramer's Rule as follows.
  If a system of n linear equations in n variables has a coefficient matrix A with a nonzero determinant, the solutions of the system are of the form

  $$x_i = \frac{|A_i|}{|A|},$$

  where A_i is formed from A by deleting the ith column and replacing it with the column of constants.

Example 2. Use Cramer's Rule solve the system of linear equations.

$$\begin{cases} 3x - 2y + 4z = 0 \\ 2x - 8y \qquad = 2 \\ 4x - 3y - 5z = 1 \end{cases} \Rightarrow D = \begin{vmatrix} 3 & -2 & 4 \\ 2 & -8 & 0 \\ 4 & -3 & -5 \end{vmatrix} = 204$$

$$D_x = \begin{vmatrix} 0 & -2 & 4 \\ 2 & -8 & 0 \\ 1 & -3 & -5 \end{vmatrix} = -12 \Rightarrow x = \frac{D_x}{D} = \frac{-12}{204} = -\frac{1}{17}$$

$$D_y = \begin{vmatrix} 3 & 0 & 4 \\ 2 & 2 & 0 \\ 4 & 1 & -5 \end{vmatrix} = -54 \Rightarrow y = \frac{D_y}{D} = \frac{-54}{204} = -\frac{9}{34}$$

$$D_z = \begin{vmatrix} 3 & -2 & 0 \\ 2 & -8 & 2 \\ 4 & -3 & 1 \end{vmatrix} = -18 \Rightarrow z = \frac{D_z}{D} = \frac{-18}{204} = -\frac{3}{34}$$

II. Area of a Triangle (p. 622) Pace: 5 minutes

- State that the area of the triangle with vertices (x_1, y_1), (x_2, y_2), and (x_3, y_3) is

$$\text{Area} = \left| \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \right|$$

Example 3. Find the area of the triangle with vertices (1, −1), (2, 3), and (5, −3).

$$\text{Area} = \left| \frac{1}{2} \begin{vmatrix} 1 & -1 & 1 \\ 2 & 3 & 1 \\ 5 & -3 & 1 \end{vmatrix} \right| = 9$$

III. Lines in a Plane (pp. 623−624) Pace: 5 minutes

- State that if the area in the preceding formula equals zero, then the three points must be collinear. State this as the **test for collinear points.**
  The three points (x_1, y_1), (x_2, y_2), and (x_3, y_3) are collinear if and only if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0 .$$

Example 4. Determine whether the points (0, 1), (4, 4), and (8, 7) are collinear.

$$\begin{vmatrix} 0 & 1 & 1 \\ 4 & 4 & 1 \\ 8 & 7 & 1 \end{vmatrix} = 0 .$$

Yes

- State that if we replace one of the points with the variables (x, y), then this formula becomes the equation of a line.

Example 5. Find the equation of the line that passes through the points (5, 2) and (−1, 0).

$$\begin{vmatrix} x & y & 1 \\ 5 & 2 & 1 \\ -1 & 0 & 1 \end{vmatrix} = 0$$

$$x\begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} - y\begin{vmatrix} 5 & 1 \\ -1 & 1 \end{vmatrix} + 1\begin{vmatrix} 5 & 2 \\ -1 & 0 \end{vmatrix} = 0$$

$$2x - 6y + 2 = 0$$

IV. Cryptography (pp. 625−627) Pace: 5 minutes

- State that we are going to use matrices to encode and decode messages. We start by assigning a number to each letter in the alphabet, starting with 0 for a space, 1 for A, and so on, up to 26 for Z.

Example 6. Use the matrix

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -2 \\ 0 & 1 & 1 \end{bmatrix}$$

to encode the message STUDY MATHEMATICS.

First we convert the message into 1 × 3 matrices.

[19 20 21] [4 25 0] [13 1 20] [8 5 13] [1 20 9] [3 19 0]

S T U D Y M A T H E M A T I C S

Next we multiply our 1 × 3 matrices by *A*.

$$\begin{bmatrix} 19 & 20 & 21 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -2 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 59 & 41 & -38 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 25 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -2 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 54 & 25 & -54 \end{bmatrix}$$

$$\begin{bmatrix} 13 & 1 & 20 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -2 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 15 & 21 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 8 & 5 & 13 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -2 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 18 & 18 & -5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 20 & 9 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -2 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 41 & 29 & -32 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 19 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -2 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 41 & 19 & -41 \end{bmatrix}$$

The encoded message is

59 41 –38 54 25 –54 15 21 5 18 18 –5 41 29 –32 41 19 –41

To decode the message, the recipient must have the inverse matrix to multiply the coded message by.