Chapter 9 Sequences, Series, and Probability

Course/Section
Lesson Number
Date

Section 9.4 Mathematical Induction

Section Objectives: Students will know how to use mathematical induction to prove a statement involving a positive integer n.

I. Introduction (pp. 673−677)   Pace: 20 minutes

- State the principle of mathematical induction as follows: Let P_n be a statement involving the positive integer n. If
  1. P_1 is true, and
  2. anytime P_k is true P_{k+1} is also true for every positive integer k,
  then P_n must be true for all positive integers n.

  Example 1. Use mathematical induction to prove the following formula.

  $$S_n = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

  1. $1 = 1(1+1)/2$. Therefore the formula is valid for $n = 1$.
  2. Assume that the formula is valid for $n = k$, and then show it is valid for $n = k + 1$.

  $$S_k = 1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}$$

  $$\Rightarrow 1 + 2 + 3 + \cdots + k + (k+1) = \frac{k(k+1)}{2} + (k+1)$$

  $$\Rightarrow S_{k+1} = \frac{k(k+1)}{2} + (k+1)$$

  $$= (k+1)\left(\frac{k}{2} + 1\right)$$

  $$= (k+1)\left(\frac{k}{2} + \frac{2}{2}\right)$$

  $$= (k+1)\left(\frac{k+2}{2}\right)$$

  $$= \frac{(k+1)((k+1)+1)}{2}$$

  Therefore the formula is valid for all positive integers n.

Tip: Slowly go through this example and show the students how the two parts of the proof cause the "domino effect." Say that the two parts of the proof immediately tell us that the formula is valid for $n = 2$. This fact, along with part 2 of the proof, tells us that the formula is valid for $n = 3$. This fact, along with part 2 of the proof, tells us that the formula is valid for $n = 4$, and so on.

Example 2. Use mathematical induction to prove the following formula

$$S_n = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

1. $1^3 = \frac{1^2(1+1)^2}{4}$. Therefore the formula is valid for $n = 1$.
2. Assume that the formula is valid for $n = k$, and then show it is valid for $n = k + 1$.

$$S_k = 1^3 + 2^3 + 3^3 + \cdots + k^3 = \frac{k^2(k+1)^2}{4}$$

$$\Rightarrow 1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3 = \frac{k^2(k+1)^2}{4} + (k+1)^3$$

$$\Rightarrow S_{k+1} = \frac{k^2(k+1)^2}{4} + (k+1)^3$$

$$= (k+1)^2\left[\frac{k^2}{4} + (k+1)\right]$$

$$= (k+1)^2\left[\frac{k^2}{4} + \frac{4(k+1)}{4}\right]$$

$$= (k+1)^2\left[\frac{k^2+4k+4}{4}\right]$$

$$= \frac{(k+1)^2(k+2)^2}{4}$$

$$= \frac{(k+1)^2((k+1)+1)^2}{4}$$

Therefore the formula is valid for all positive integers n.

II. Pattern Recognition (pp. 677−678) Pace: 5 minutes

- State that sometimes we do not have a formula that applies. In this case, we have to produce a formula through pattern recognition and then prove it by mathematical induction.

Example 3. Find a formula for the nth partial sum of

$$a_n = \frac{1}{(n+2)(n+3)}.$$

$$S_1 = \frac{1}{3\cdot 4} = \frac{1}{3(1+3)}$$

$$S_2 = \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} = \frac{2}{15} = \frac{2}{3(2+3)}$$

$$S_3 = \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \frac{1}{5\cdot 6} = \frac{3}{18} = \frac{3}{3(3+3)}$$

$$S_4 = \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \frac{1}{5\cdot 6} + \frac{1}{6\cdot 7} = \frac{4}{21} = \frac{4}{3(4+3)}$$

So, it appears that $S_n = \frac{n}{3(n+3)}$. We will prove this by mathematical induction. We already know that S_1 is valid. Next we will assume that S_k is valid and try to prove that S_{k+1} is valid.

$$S_k = \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \frac{1}{5 \cdot 6} + \cdots + \frac{1}{(k+2)(k+3)} = \frac{k}{3(k+3)}$$

$$\Rightarrow \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \frac{1}{5 \cdot 6} + \cdots + \frac{1}{(k+2)(k+3)} + \frac{1}{((k+1)+2)((k+1)+3)}$$

$$= \frac{k}{3(k+3)} + \frac{1}{((k+1)+2)((k+1)+3)} \Rightarrow$$

$$S_{k+1} = \frac{k}{3(k+3)} + \frac{1}{((k+1)+2)((k+1)+3)}$$

$$= \frac{k}{3(k+3)} + \frac{1}{(k+3)(k+4)}$$

$$= \frac{k^2 + 4k + 3}{3(k+3)(k+4)}$$

$$= \frac{(k+1)(k+3)}{3(k+3)(k+4)}$$

$$= \frac{(k+1)}{3((k+1)+3)}$$

Therefore the formula is valid for all positive integers *n*.

III. Sums of Powers of Integers (p. 679) Pace: 10 minutes

- State the following formulas for the **sums of powers of integers.**

1. $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$
2. $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$
3. $1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$
4. $1^4 + 2^4 + 3^4 + \cdots + n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$
5. $1^5 + 2^5 + 3^5 + \cdots + n^5 = \frac{n^2(n+1)^2(2n^2+2n-1)}{12}$

Example 4. Find the following sums.

a) $$\sum_{n=1}^{10} n^5 = \frac{10^2(10+1)^2(2 \cdot 10^2 + 2 \cdot 10 - 1)}{12} = 220,825$$

b) $$\sum_{n=1}^{20} (5n^2 - 7n) = 5\sum_{n=1}^{20} n^2 - 7\sum_{n=1}^{20} n$$

$$= 5 \cdot \frac{20(20+1)(2 \cdot 20 + 1)}{6} - 7 \cdot \frac{20(20+1)}{2} = 12,880$$

IV. Finite Differences (p. 680) Pace: 5 minutes

- State that **first differences** of a sequence are found by subtracting consecutive terms. **Second differences** are found by subtracting consecutive first differences. Also, if the second differences are all the same, then the sequence has a perfect quadratic model.

Example 5. Find a model for the sequence 5, 10, 17, 26, 37, 50,…

5	10	17	26	37	50
5	7	9	11	13	
2	2	2	2		

Since the second differences are all 2, the sequence has a quadratic model, $a_n = an^2 + bn + c$.

$a_1 = a + b + c = 5$
$a_2 = 4a + 2b + c = 10$
$a_3 = 9a + 3b + c = 17$

Solving this system of equations, we get $a_n = n^2 + 2n + 2$.