Chapter 9 Sequences, Series, and Probability

Course/Section Lesson Number Date

Section 9.6 Counting Principles

Section Objectives: Students will know how to solve counting problems using the Fundamental Counting Principle, permutations, and combinations.

I. Simple Counting Problems (p. 691) Pace: 5 minutes

Example 1. An urn contains six balls, numbered 1 through 6. A ball is drawn from the urn, its number noted, and then it is placed back into the urn. A second ball is drawn and its number noted. In how many different ways can a total of 8 be obtained from the two balls?

Consider the following ordered pairs to be of the form
(*first number*, *second number*).
(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)
So, there are five different ways to obtain a total of 8.

Example 2. An urn contains six balls, numbered 1 through 6. A ball is drawn from the urn and its number noted, but it is *not* placed back into the urn. A second ball is drawn and its number noted. In how many different ways can a total of 8 be obtained from the two balls?

Consider the following ordered pairs to be of the form
(*first number*, *second number*).
(2, 6), (3, 5), (5, 3), (6, 2)
So, there are four different ways to obtain a total of 8.

- State that the key difference between the above examples is **replacement versus no replacement.**

II. The Fundamental Counting Principle (p. 692) Pace: 10 minutes

- State that some events can occur in so many different ways that it is impractical to list them all. In these cases we need to rely on formulas. Our first is the **Fundamental Counting Principle**, which states:
  Let E_1 and E_2 be two events. The first event E_1 can occur in m_1 different ways. After E_1 has occurred, E_2 can occur in m_2 different ways. The number of ways that the two events can occur is $m_1 \bullet m_2$. Note that this principle can be extended to any finite number of events.

Example 3. How many three-letter triplets can be formed from the English alphabet?
$26^3 = 17{,}576$

III. Permutations (pp. 693−695) Pace: 15 minutes

- Define a **permutation** of n different elements as an ordering of the elements such that one element is first, one is second, and so on.

Example 4. How many permutations of the letters A, B, C, and D are possible?
There are four "slots" to fill. The first can be filled in four different ways, the second in three ways, the third in two ways, and the last in one way.
$4! = 24$.

Example 5. How many permutations are possible from just two of the letters A, B, C, and D?
There are two "slots" to fill. The first can be filled in four different ways, and the second in three ways.
$4 \bullet 3 = 12$.

- This is an example of the following formula.
The number of permutations of n elements taken r at a time is
$${}_nP_r = \frac{n!}{(n-r)!} = n(n-1)(n-2)\cdots(n-r+1).$$

- The above formula is used when all the elements are unique. If not all the elements are unique, then not all the permutations are distinguishable. In this case, we use the following.
Suppose a set of n objects has n_1 of one kind of object, n_2 of a second kind, and so on, with $n = n_1 + n_2 + \cdots + n_k$. Then the number of **distinguishable permutations** of the n objects is
$$\frac{n!}{n_1!\cdot n_2!\cdots n_k!}.$$

Example 6. In how many distinguishable ways can the letters in the word MATHEMATICS be written?
Since there are two M's, two A's, and two T's,
$$\frac{11!}{2!\cdot 2!\cdot 2!\cdot 1!\cdot 1!\cdot 1!\cdot 1!\cdot 1!} = 4,989,600.$$

IV. Combinations (pp. 696−697) Pace: 10 minutes

- Describe a permutation as selecting, from a club of 40 people, a president, a treasurer, and a secretary. Contrast this with selecting, from a club of 40 people, a committee of 3 people. The latter is an example of a **combination.**

Tip: It should be emphasized that with a permutation, order is important and with a combination, it is not.

- State that the number of combinations of n elements taken r at a time is
$${}_nC_r = \frac{n!}{r!(n-r)!}.$$

Example 7. In how many different ways can a committee of 3 be selected from a club of 40?
${}_{40}C_3 = 9,880$

Example 8. How many subsets of a set of four elements can be formed?
Number of subsets with 0 elements = ${}_4C_0$
Number of subsets with 1 elements = ${}_4C_1$
Number of subsets with 2 elements = ${}_4C_2$
Number of subsets with 3 elements = ${}_4C_3$
Number of subsets with 4 elements = ${}_4C_4$
Summing these, the result is the same as the binomial expansion of $(1 + 1)^4 = 2^4 = 16$.